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Multiplicity Result for a Degenerate Elliptic Equation with Logistic Reaction[1]

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A degenerate elliptic equation $\lambda\,\Delta_p u + u^{q-1}(1-u^r)=0$ with zero Dirichlet boundary condition, where λ is a positive parameter, $2<p<q$, and $r>0$, is studied.

for sufficiently small λ $(<\Lambda)$, the problem has at least two solutions. In this article, it is shown that if $\lambda<\Lambda$, then there exist at least two positive solutions, via the variational method.

Key Words: p-Laplace operator; degenerate elliptic equation; multiple solutions; mountain pass theorem.

1. INTRODUCTION AND RESULTS

Let Ω be a connected, bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, with $C^{2,\alpha}$-boundary $\partial\Omega$ for some $\alpha \in (0,1)$. We consider the following degenerate elliptic equation,

$$(\mathrm{P})_\lambda \begin{cases} \lambda\,\Delta_p u + f(u) = 0 & \text{in } \Omega, \\ u \geqslant 0,\ \not\equiv 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where λ is a positive parameter, Δ_p is the p-Laplace operator given by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\,\nabla u)$, and f is given by

$$f(u) = u^{q-1}(1-u^r)$$

with $2<p<q$ and $r>0$.

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The problem $(\mathrm{P})_\lambda$ has been studied by many authors in the case $p \geqslant q > 2$ (cf. García-Melián and Sabina de Lis [7], Guedda and Véron [8], Kamin and Véron [9], Takeuchi and Yamada [17]). In this case, since $f(s)/s^{p-1}$ becomes monotone decreasing for $s \in (0, 1]$, we know that the solution of $(\mathrm{P})_\lambda$ is unique (as far as it exists) from the uniqueness result of Díaz and Saa [5]. It is also known that the unique solution has *flat core property*; that is, for sufficiently small $\lambda > 0$, there appears a flat place in the graph of solution, and the uniqueness of solution plays an important role to study the existence and λ-dependence of flat core.

For the case $2 < p < q$, the structure of solutions for $(\mathrm{P})_\lambda$ is different. For $N = 1$, the author and Yamada [17, Theorem 3.3] have obtained complete information on the global bifurcation structure of solutions of $(\mathrm{P})_\lambda$ and concluded that there exist exactly two solutions for a certain range of λ by the phase-plane analysis (we have also studied in [17] the non-stationary problem associated with $(\mathrm{P})_\lambda$; see also [15]). However, in higher dimensional case, phase-plane analysis is no longer useful and one has to approach by other methods. Recently, the author [16] has proved the following results for $N \geqslant 2$:

THEOREM 1.1 [16]. *Let $2 < p < q$ and $r > 0$. Then there exists a positive number Λ such that*

(i) *if $\lambda > \Lambda$, then $(\mathrm{P})_\lambda$ has no solution*;

(ii) *if $\lambda \leqslant \Lambda$, then $(\mathrm{P})_\lambda$ has a maximal solution $\bar{u}_\lambda$.*

Furthermore, there exists a positive number $\Lambda^* \in (0, \Lambda]$ such that for any $\lambda \in (0, \Lambda^*)$, $(\mathrm{P})_\lambda$ has another solution $u_\lambda \leqslant \bar{u}_\lambda$, $\not\equiv \bar{u}_\lambda$.

Remark 1.1. (i) Non-existence of solution has been proved by Véron [19] for the p-Laplace operator on a compact Riemannian manifold without boundary.

(ii) A *maximal solution* is defined as a solution u satisfying $u \geqslant v$ for all solutions v of $(\mathrm{P})_\lambda$.

(iii) Flat core property and its λ-dependence of maximal solution have been also studied in [16].

(iv) Cañada, Drábek, and Gámez [3a] have mentioned a second solution of $(\mathrm{P})_\lambda$. However, they gave no result about it.

In the proof of Theorem 1.1, maximal solutions are constructed by the usual barrier method. As for the existence of second solution, the proof of [16] uses a variational approach in the following manner. Observe that solutions of $(\mathrm{P})_\lambda$ satisfy $0 < u \leqslant 1$ in Ω (see [16, Proposition 2.1]), so that they coincide with critical points of the following C^1-functional Φ on $W_0^{1,p}$,

$$\Phi(u) = \frac{\lambda}{p} \|\nabla u\|_p^p - \int_\Omega \bar{F}(u)\, dx, \tag{1.1}$$

where $\bar{F}(u)=\int_0^u \bar{f}_\xi(s)\,ds$ and $\bar{f}_\xi(s):=f(s)$ in $[0,\xi]$, $:=0$ in $(-\infty,0)$ and $:=f(\xi)$ in $(\xi,+\infty)$ for any $\xi>1$ fixed. Here, $\|\cdot\|_p$ denotes L^p-norm. We can show that Φ satisfies the Palais–Smale condition (cf. Dinca *et al.* [6], the proof of [16, Theorem 1.3]) and that there exists a positive number $\Lambda^*\in(0,\Lambda]$ such that for any $\lambda\in(0,\Lambda^*)$, the value of Φ to $(\mathrm{P})_\lambda$ at $\bar{u}_\lambda$ becomes negative, hence $\Phi(0)=0>\Phi(\bar{u}_\lambda)$. Thus, the usual mountain pass theorem (see Ambrosetti and Rabinowitz [2], Rabinowitz [14]) gives a second solution of $(\mathrm{P})_\lambda$. However, we had no information whether $(\mathrm{P})_\lambda$ admits a second solution for any $\lambda\in(0,\Lambda)$, that is, $\Lambda^*=\Lambda$.

In this paper, we will show the following "maximal" result for existence of second solution of $(\mathrm{P})_\lambda$, which is an extension of the last assertion of Theorem 1.1:

THEOREM 1.2. *Let $2<p<q$ and $r>0$. Then, for any $\lambda\in(0,\Lambda)$, $(\mathrm{P})_\lambda$ has a solution u_λ satisfying $u_\lambda\leqslant\bar{u}_\lambda$, $\not\equiv\bar{u}_\lambda$, where Λ is the number appearing in Theorem* 1.1 *and $\bar{u}_\lambda$ is the maximal solution of* $(\mathrm{P})_\lambda$.

In order to prove Theorem 1.2, our strategy is to apply an extended mountain pass theorem by Pucci and Serrin, which asserts that, if Φ has a pair of local minima, then Φ possesses a third critical point (see Pucci and Serrin [12, Theorem 4], also [14, Corollary 3.15]). In the next section, we will prove that the trivial solution $u=0$ is a local minimizer of Φ in $W_0^{1,p}$ for every $\lambda>0$ (Lemma 2.1), and that if the maximal solution $\bar{u}_\lambda$ is isolated, then $\bar{u}_\lambda$ is also a local minimizer of Φ in $W_0^{1,p}$ for $\lambda\in(0,\Lambda)$ (Lemmas 2.2 and 2.3). Finally we can conclude Theorem 1.2.

Remark 1.4. For the linear diffusion case $2=p<q$, Rabinowitz [13] has studied $(\mathrm{P})_\lambda$ by combining critical point theory and the Leray–Schauder degree theory, and obtained Theorem 1.2 (see also [2, 14]). Especially, when Ω is a ball, Ouyang and Shi [11] have obtained precise global bifurcation diagram and concluded that there exist *exactly* two solutions for small λ by using a bifurcation theorem of Crandall and Rabinowitz [4].

2. PROOF OF THEOREM 1.2

LEMMA 2.1. *For every $\lambda>0$, the trivial solution $u=0$ is a local minimizer of Φ in $W_0^{1,p}$.*

Proof. Since $p<q$, for any $\delta>0$ there exists $C_\delta>0$ such that $\bar{f}_\xi(s)\leqslant\delta s^{p-1}+C_\delta s^{q^*-1}$, where q^* is any number satisfying $p<q^*<p^*$ and $p^*:=Np/(N-p)$ if $p<N$, $:=+\infty$ if $p\geqslant N$. Then $\bar{F}(u)\leqslant\delta u^p/p+C_\delta u^{q^*}/q^*$. Thus, the Sobolev inequality assures

$$\Phi(u) \geqslant \frac{\lambda}{p} \|\nabla u\|_p^p - \frac{\delta}{p} \|u\|_p^p - \frac{C_\delta}{q^*} \|u\|_{q^*}^{q^*}$$

$$\geqslant \left(\frac{\lambda - C_1 \delta}{p} - \frac{C_2 C_\delta}{q^*} \|\nabla u\|_p^{q^* - p} \right) \|\nabla u\|_p^p,$$

where C_1, C_2 are positive constants and $\delta \in (0, \lambda/C_1)$. Therefore, we see that there exists a positive number ρ such that $\Phi(u) \geqslant 0 = \Phi(0)$ if $\|\nabla u\|_p \leqslant \rho$. ▮

Fix $\lambda \in (0, \Lambda)$ and let λ_i, ε_i $(i=1, 2)$ be numbers satisfying that $0 < \lambda_2 < \lambda < \lambda_1 \leqslant \Lambda$ and $(\lambda/\lambda_1)^{1/(q-p)} < \varepsilon_1 < 1 < \varepsilon_2 < \min\{\xi, (\lambda/\lambda_2)^{1/(q-p)}\}$. Then we can see that $u_1 := \varepsilon_1 \bar{u}_{\lambda_1}$ is a lower solution and $u_2 := \varepsilon_2 \bar{u}_{\lambda_2}$ is an upper solution of $(\mathrm{P})_\lambda$, respectively, where $\bar{u}_{\lambda_i}$ $(i=1, 2)$ is the maximal solution of $(\mathrm{P})_{\lambda_i}$ (for the definitions of upper and lower solution, see Section 1 in [16]). Note that $\bar{u}_\lambda$ is an interior point of

$$\mathscr{A} := \{u \in C_0^1(\bar{\Omega});\, u_1 \leqslant u \leqslant u_2 \text{ in } \Omega\}, \tag{2.1}$$

with respect to C^1-topology by the maximum principle due to Vázquez [18, Theorem 5], and that $f(u) = \bar{f}_\xi(u)$ for all $u \in \mathscr{A}$.

LEMMA 2.2. *Let $\lambda \in (0, \Lambda)$ and assume that $(\mathrm{P})_\lambda$ has no solution in $\mathscr{A}$ except for $\bar{u}_\lambda$. Then $\bar{u}_\lambda$ is a local minimizer of Φ in C_0^1.*

Proof. Let $\tilde{f}$ be the truncated function of $\bar{f}_\xi$ as

$$\tilde{f}(x, s) := \begin{cases} \bar{f}_\xi(u_1(x)) & \text{if } \; s < u_1(x), \\ \bar{f}_\xi(s) & \text{if } \; u_1(x) \leqslant s \leqslant u_2(x), \\ \bar{f}_\xi(u_2(x)) & \text{if } \; s > u_2(x) \end{cases}$$

and set $\tilde{F}(x, u) := \int_0^u \tilde{f}(x, s)\, ds$. Using $\tilde{F}$, we consider the following auxiliary functional $\tilde{\Phi}$ associated with Φ:

$$\tilde{\Phi}(u) = \frac{\lambda}{p} \|\nabla u\|_p^p - \int_\Omega \tilde{F}(x, u)\, dx.$$

It follows from the direct method that $\tilde{\Phi}$ has a global minimizer $u_0 \in W_0^{1,p}$. Therefore u_0 satisfies

$$\lambda\, \Delta_p u_0 + \tilde{f}(x, u_0) = 0 \qquad \text{in } \Omega \tag{2.2}$$

and we see $u_0 \in C^1_0(\bar{\Omega})$ by Lieberman's regularity result [10]. Moreover, since u_1 is a (weak) lower solution of $(P)_\lambda$ and u_0 is a (weak) solution of (2.2),

$$\begin{aligned}\lambda \int_{\{u_1 > u_0\}} & (|\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_0|^{p-2} \nabla u_0) \cdot (\nabla u_1 - \nabla u_0)\, dx \\ &\leqslant \int_{\{u_1 > u_0\}} (f(u_1) - \tilde{f}(x, u_0))(u_1 - u_0)\, dx \\ &= \int_{\{u_1 > u_0\}} (f(u_1) - \bar{f}_\xi(u_1))(u_1 - u_0)\, dx = 0. \qquad (2.3)\end{aligned}$$

Here we have used the function $\max\{u_1 - u_0, 0\} \in W^{1,p}_0(\Omega)$ as test function. The left-hand side of (2.3) is bounded from below by

$$C_0 \int_{\{u_1 > u_0\}} |\nabla(u_1 - u_0)|^p\, dx;$$

so that $\{u_1 > u_0\} = \varnothing$ and we obtain $u_1 \leqslant u_0$ in Ω. It also follows from similar arguments that $u_0 \leqslant u_2$ in Ω (note that $f(u_2) = \bar{f}_\xi(u_2)$ since $u_2 \leqslant \xi$). Therefore, $u_0 \in \mathscr{A}$ and (2.2) becomes $\lambda\, \Delta_p u_0 + \bar{f}_\xi(u_0) = 0$ in Ω; consequently u_0 is a solution of $(\mathrm{P})_\lambda$, which belongs to $\mathscr{A}$. By the assumption, $u_0 = \bar{u}_\lambda$, hence $\bar{u}_\lambda$ is a global minimizer of $\tilde{\Phi}$ in $W^{1,p}_0$.

Now, if $\varepsilon > 0$ is sufficiently small, then any $u \in C^1_0(\bar{\Omega})$ with $\|u - \bar{u}_\lambda\|_{C^1} < \varepsilon$ satisfies $u \in \mathscr{A}$ because $\bar{u}_\lambda$ is an interior point of $\mathscr{A}$. Furthermore, for any $u \in \mathscr{A}$

$$\begin{aligned}\Phi(u) - \tilde{\Phi}(u) &= \int_\Omega \int_0^{u(x)} (\bar{f}_\xi(s) - \tilde{f}(x, s))\, ds\, dx \\ &= \int_\Omega \int_0^{u_1(x)} (\bar{f}_\xi(s) - \bar{f}_\xi(u_1(x)))\, ds\, dx\end{aligned}$$

is a constant independent of u. Since $\bar{u}_\lambda$ is a global minimizer of $\tilde{\Phi}$, it consequently becomes a local minimizer of Φ in C^1_0.

Remark 2.1. The proof of Lemma 2.2 is essentially due to Brézis and Nirenberg [3] (see also Ambrosetti *et al.* [1]).

LEMMA 2.3. *Let $\lambda \in (0, \Lambda)$ and assume that $(\mathrm{P})_\lambda$ admits no solution in $\mathscr{A}$ except for $\bar{u}_\lambda$. Then $\bar{u}_\lambda$ is a local minimizer of Φ in $W^{1,p}_0$.*

Proof. Suppose that for any neighborhood O of $\bar{u}_\lambda$ in $W^{1,p}_0$, there exists $v \in O$ such that $\Phi(v) < \Phi(\bar{u}_\lambda)$. Then, for sufficiently small $\varepsilon > 0$ there exists $v_\varepsilon \in B_\varepsilon$ such that $\Phi(v_\varepsilon) < \Phi(\bar{u}_\lambda)$, where $B_\varepsilon := \{u \in W^{1,p}_0(\Omega);\ \|u - \bar{u}_\lambda\|_2 \leqslant \varepsilon\}$, because Sobolev's inequality allows us to take a neighborhood $O \subset B_\varepsilon$ of $\bar{u}_\lambda$

in $W_0^{1,p}$. Moreover, we may assume that v_ε is a global minimizer of Φ in B_ε without loss of generality.

If $\|v_\varepsilon - \bar{u}_\lambda\|_2 < \varepsilon$, then v_ε becomes a local minimizer of Φ in $W_0^{1,p}$, hence v_ε is a solution of $(\mathrm{P})_\lambda$ and $0 < v_\varepsilon \leqslant 1$. We next consider the case $\|v_\varepsilon - \bar{u}_\lambda\|_2 = \varepsilon$. Define $K(u) = \|u - \bar{u}_\lambda\|_2^2/2$. Then there exists Lagrange's multiplier μ_ε such that $\Phi'(v_\varepsilon) = \mu_\varepsilon K'(v_\varepsilon)$, i.e.,

$$\lambda \int_\Omega |\nabla v_\varepsilon|^{p-2} \nabla v_\varepsilon \cdot \nabla \zeta \, dx - \int_\Omega \bar{f}_\xi(v_\varepsilon) \, \zeta \, dx = \mu_\varepsilon \int_\Omega (v_\varepsilon - \bar{u}_\lambda) \, \zeta \, dx \qquad (2.4)$$

for all $\zeta \in W_0^{1,p}$. In order to show $\mu_\varepsilon \leqslant 0$, suppose $\mu_\varepsilon > 0$. Then there exists $h \in W_0^{1,p}$ such that $\langle \Phi'(v_\varepsilon), h \rangle < 0$ and $\langle K'(v_\varepsilon), h \rangle < 0$, where $\langle \cdot, \cdot \rangle$ denotes the duality product between $W_0^{1,p}$ and its dual space. Since Taylor's theorem gives $K(v_\varepsilon + \tau h) = K(v_\varepsilon) + \tau \langle K'(v_\varepsilon), h \rangle + o(\tau)$, we get $K(v_\varepsilon + \tau h) < \varepsilon^2/2$ for sufficiently small $\tau > 0$, and hence $v_\varepsilon + \tau h \in B_\varepsilon$. In the same way, we have $\Phi(v_\varepsilon + \tau h) < \Phi(v_\varepsilon)$. These facts mean that v_ε is not a global minimizer of Φ in B_ε, which is a contradiction. Thus we have shown $\mu_\varepsilon \leqslant 0$. Now, we will return (2.4), i.e., $\lambda \Delta_p v_\varepsilon + g(\mu_\varepsilon, x, v_\varepsilon) = 0$, where $g(a, x, s) := \bar{f}_\xi(s) + a(s - \bar{u}_\lambda(x))$. Noting $\bar{u}_\lambda \leqslant 1$, we can observe that $g(a, x, s) \geqslant 0$ in $\{a \leqslant 0\} \times \Omega \times \{s \leqslant 0\}$ and $g(a, x, s) \leqslant 0$ in $\{a \leqslant 0\} \times \Omega \times \{s \geqslant 1\}$. These facts assure that $0 \leqslant v_\varepsilon \leqslant 1$. Moreover, there exists a number $M > 0$ such that M is independent of ε and $\mu_\varepsilon \geqslant -M$. Indeed, suppose that $\mu_\varepsilon \to -\infty$ as $\varepsilon \to 0$. Then, for sufficiently small $\varepsilon > 0$, $g(\mu_\varepsilon, x, s)$ is decreasing in s and (2.4) has a unique solution $v_\varepsilon = \bar{u}_\lambda$, which contradicts $\Phi(v_\varepsilon) < \Phi(\bar{u}_\lambda)$. Therefore, in any case, Lieberman's regularity result [10] yields $\|v_\varepsilon\|_{C^{1,\beta}} \leqslant C$ for some constants $C > 0$ and $\beta \in (0, 1)$ independent of ε. Thus, the Ascoli–Arzelà theorem allows us to take a subsequence $\{v_{\varepsilon'}\}$ of $\{v_\varepsilon\}$ satisfying $v_{\varepsilon'} \to \bar{u}_\lambda$ in C^1 (here, we have used $v_{\varepsilon'} \in B_{\varepsilon'}$). This result, together with $\Phi(v_{\varepsilon'}) < \Phi(\bar{u}_\lambda)$, contradicts Lemma 2.2. ∎

Remark 2.2. Brézis and Nirenberg [3] have shown that for a certain functional corresponding to semilinear elliptic equations, its local minimizer in C^1 becomes a local minimizer in $H^1 = W^{1,2}$. Lemma 2.3 is a partial extension of [3] to $W^{1,p}$ versus C^1.

Proof of Theorem 1.2. Define Φ as (1.1). As mentioned in Introduction, Φ satisfies the Palais–Smale condition. Let $\mathscr{A}$ be the set defined by (2.1). If there exists a solution distinct from $\bar{u}_\lambda$ in $\mathscr{A}$, then we have nothing to prove. Thus we may assume that there exists no solution in $\mathscr{A}$ except for $\bar{u}_\lambda$. Then, from Lemmas 2.1 and 2.3, we have obtained two local minimizers 0 and $\bar{u}_\lambda$ of Φ in $W_0^{1,p}$. Therefore, it follows from an extended mountain pass theorem by Pucci and Serrin [12, Theorem 4] (see also [14, Corollary 3.15]) that there exists a third critical point of Φ, which is a solution of $(\mathrm{P})_\lambda$ distinct from 0 and $\bar{u}_\lambda$. ∎

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REFERENCES

1. A. Ambrosetti, H. Brézis, and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.* **122** (1994), 519–543.
2. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.* **14** (1973), 349–381.
3. H. Brézis and L. Nirenberg, H^1 versus C^1 local minimizers, *C. R. Acad. Sci. Paris Sér. I Math.* **317** (1993), 465–472.

3a. A. Cañada, P. Drábek, and J. L. Gámez, Existence of positive solutions for some problems with nonlinear diffusion, *Trans. Amer. Math. Soc.* **349** (1997), 4231–4249.

4. M. G. Crandall and P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, *Arch. Rational Mech. Anal.* **52** (1973), 161–180.
5. J. I. Díaz and J. E. Saa, Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires, *C. R. Acad. Sci. Paris Sér. I Math.* **305** (1987), 521–524.
6. G. Dinca, P. Jebelean, and J. Mawhin, A result of Ambrosetti–Rabinowitz type for p-Laplacian, *in* "Qualitative Problems for Differential Equations and Control Theory," pp. 231–242, World Scientific, River Edge, 1995.
7. J. García-Melián and J. Sabina de Lis, Stationary profiles of degenerate problems when a parameter is large, *Differential Integral Equations* **13** (2000), 1201–1232.
8. M. Guedda and L. Véron, Bifurcation phenomena associated to the p-Laplace operator, *Trans. Amer. Math. Soc.* **310** (1988), 419–431.
9. S. Kamin and L. Véron, Flat core properties associated to the p-Laplace operator, *Proc. Amer. Math. Soc.* **118** (1993), 1079–1085.
10. G. M. Lieberman, Boundary regularity for solutions of degenerate elliptic equations, *Nonlinear Anal.* **12** (1988), 1203–1219.
11. T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problems, *J. Differential Equations* **146** (1998), 121–156.
12. P. Pucci and J. Serrin, Extensions of the mountain pass theorem, *J. Funct. Anal.* **59** (1984), 185–210.
13. P. H. Rabinowitz, Pairs of positive solutions of nonlinear elliptic partial differential equations, *Indiana Math. J.* **23** (1973), 172–185.
14. P. H. Rabinowitz, "Minimax Methods in Critical Point Theory with Applications to Differential Equations," CBMS Regional Conf. Ser. in Math., Vol. 65, Amer. Math. Soc., Providence, 1986.
15. S. Takeuchi, Behavior of solutions near the flat hats of stationary solutions for a degenerate parabolic equation, *SIAM J. Math. Anal.* **31** (2000), 678–692.
16. S. Takeuchi, Positive solutions of a degenerate elliptic equation with logistic reaction, *Proc. Amer. Math. Soc.* **129** (2001), 433–441.
17. S. Takeuchi and Y. Yamada, Asymptotic properties of a reaction-diffusion equation with degenerate p-Laplacian, *Nonlinear Anal.* **42** (2000), 41–61.
18. J. L. Vázquez, A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12** (1984), 191–202.
19. L. Véron, Première valeur propre non nulle du p-laplacien et équations quasi linéaires elliptiques sur une variété riemannienne compacte, *C. R. Acad. Sci. Paris Sér. I Math.* **314** (1992), 271–276.